

ATOMIC BLOCKS FOR NONCOMMUTATIVE MARTINGALES

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ABSTRACT. Given a probability space (Ω, Σ, μ) , the Hardy space $H_1(\Omega)$ which is associated to the martingale square function does not admit a classical atomic decomposition when the underlying filtration is not regular. In this paper we construct a decomposition of $H_1(\Omega)$ into ‘atomic blocks’ in the spirit of Tolsa, which we will introduce for martingales. We provide three proofs of this result. Only the first one also applies to noncommutative martingales, the main target of this paper. The other proofs emphasize alternative approaches for commutative martingales. One might be well-known to experts, using a weaker notion of atom and approximation by atomic filtrations. The last one adapts Tolsa’s argument replacing medians by conditional medians.

Introduction

Let (Ω, Σ, μ) be a probability space equipped with a filtration $(\Sigma_k)_{k \geq 1}$ whose union generates Σ . Let us write E_k to denote the conditional expectation onto Σ_k -measurable functions and $\Delta_k = E_k - E_{k-1}$ for the associated differences, with the convention that $\Delta_1 = E_1$. Given $f \in L_1(\Omega)$, we shall usually write f_k and df_k for $E_k f$ and $\Delta_k f$ respectively. Once the filtration $(\Sigma_k)_{k \geq 1}$ is fixed, the martingale Hardy space $H_1(\Omega)$ is the subspace of functions f in $L_1(\Omega)$ whose $H_1(\Omega)$ -norm defined below is finite

$$\|f\|_{H_1(\Omega)} = \left\| \left(\sum_{k \geq 1} |df_k|^2 \right)^{\frac{1}{2}} \right\|_1.$$

As it was proved by Davis [6], we obtain an equivalent norm after replacing the martingale square function above by Doob’s martingale maximal function. On the contrary, replacing the martingale square function by its conditioned form we get the so-called little Hardy space $h_1(\Omega)$. In other words, the subspace of functions f in $L_1(\Omega)$ whose $h_1(\Omega)$ -norm below is finite under the convention $E_{k-1}|df_k|^2 = |f_1|^2$ when $k = 1$

$$\|f\|_{h_1(\Omega)} = \left\| \left(\sum_{k \geq 1} E_{k-1} |df_k|^2 \right)^{\frac{1}{2}} \right\|_1.$$

Both spaces are fair generalizations of the Euclidean Hardy space. Namely, if we pick the standard dyadic filtration in \mathbb{R}^n , it turns out that $H_1(\Omega)$ is by all means the dyadic form of H_1 , whereas we have $h_1(\Omega) \simeq H_1(\Omega)$ for regular filtrations as it happens in the dyadic setting. It is in the case of nonregular filtrations when both spaces have their own identity. In general, we have $h_1(\Omega) \subsetneq H_1(\Omega)$ and more precisely we know from [16] that

$$\|f\|_{H_1(\Omega)} \sim \inf_{f=g+h} \|g\|_{h_1(\Omega)} + \sum_{k \geq 1} \|dh_k\|_1.$$

We refer to Garsia’s book [8] for more information on martingale Hardy spaces.

Given $1 < p \leq \infty$, a function $a : \Omega \rightarrow \mathbb{C}$ is called a martingale p -atom when a is Σ_1 -measurable and $\|a\|_1 = 1$ or there exists $k \geq 1$ and a measurable set $A \in \Sigma_k$ such that

- $E_k(a) = 0$,
- $\text{supp}(a) \subset A$,
- $\|a\|_p \leq \mu(A)^{-\frac{1}{p'}}$ for $\frac{1}{p} + \frac{1}{p'} = 1$.

The motivation for this article is the fact that (with this notion of atom) no atomic description is known for the space $H_1(\Omega)$. On the contrary, $h_1(\Omega)$ always admits an atomic decomposition. Indeed, define the atomic Hardy spaces as

$$\begin{aligned} h_{\text{at}}^1(\Omega) &= \left\{ f \in L_1(\Omega) \mid f = \sum_{j \geq 1} \lambda_j a_j, \text{ } a_j \text{ 2-atom, } \sum_{j \geq 1} |\lambda_j| < \infty \right\}, \\ h_{\text{at},p}^1(\Omega) &= \left\{ f \in L_1(\Omega) \mid f = \sum_{j \geq 1} \lambda_j a_j, \text{ } a_j \text{ } p\text{-atom, } \sum_{j \geq 1} |\lambda_j| < \infty \right\}. \end{aligned}$$

The norm is the infimum of $\sum_j |\lambda_j|$ over all decompositions of f . As a combination of [9, 28], we know that $h_1(\Omega) \simeq h_{\text{at},p}^1(\Omega)$ for $1 < p \leq \infty$. This yields an atomic decomposition of $h_1(\Omega)$, which works for $H_1(\Omega)$ when the filtration is regular.

Atomic decompositions are useful to provide endpoint estimates for singular operators T failing to be bounded in $L_1(\Omega)$. Indeed, this typically reduces —under mild regularity assumptions— to bound uniformly the L_1 -norm of $T(a)$ for an arbitrary atom a , which is easier than proving the $H_1 \rightarrow L_1$ boundedness of T due to the particular structure of atoms. The drawback of the martingale atoms described above is that they are useless for $H_1(\Omega)$ when the filtration is not regular. This is significant because in that case the spaces $h_1(\Omega)$ are not endpoint interpolation spaces in the L_p scale, whereas the spaces $H_1(\Omega)$ are. Therefore, the goal of this paper is to provide an alternative atomic decomposition for $H_1(\Omega)$ suitable for arbitrary filtrations, and also for classical and noncommutative martingales.

Our approach is strongly motivated by the work of Tolsa on the so-called RBMO spaces [26]. Namely, it is well-known that we have $h_1(\Omega)^* \simeq \text{bmo}(\Omega)$ and also $H_1(\Omega)^* \simeq \text{BMO}(\Omega)$ where both martingale BMO spaces are respectively defined as the functions f in $L_2(\Omega)$ with finite norm

$$\begin{aligned} \|f\|_{\text{bmo}(\Omega)} &= \sup_{k \geq 1} \left\| \left(E_k |f - E_k f|^2 \right)^{\frac{1}{2}} \right\|_{\infty}, \\ \|f\|_{\text{BMO}(\Omega)} &= \sup_{k \geq 1} \left\| \left(E_k |f - E_{k-1} f|^2 \right)^{\frac{1}{2}} \right\|_{\infty}. \end{aligned}$$

It is easily checked that we have the norm equivalence

$$\|f\|_{\text{BMO}(\Omega)} \simeq \|f\|_{\text{bmo}(\Omega)} + \sup_{k \geq 1} \|df_k\|_{\infty}.$$

In analogy, Tolsa's RBMO norm is the sum of a 'doubling' BMO norm plus a term which measures the 'distance' between averages over nested pairs of doubling cubes. This viewpoint is fruitful in both directions. Indeed, nondoubling techniques are adapted here (in one of the approaches we follow) for martingales whereas martingale techniques are used in [4] for nondoubling spaces.

Tolsa's construction of the predual of RBMO is therefore our model to produce an atomic type decomposition of $H_1(\Omega)$. A Σ -measurable function $b : \Omega \rightarrow \mathbb{C}$ will be called a *martingale p -atomic block* when $b \in L_1(\Omega, \Sigma_1, \mu)$ or there exists $k \geq 1$ such that the following properties hold

- $E_k(b) = 0$,
- $b = \sum_j \lambda_j a_j$ where
 - $\text{supp}(a_j) \subset A_j$,
 - $\|a_j\|_p \leq \mu(A_j)^{-\frac{1}{p'}} \frac{1}{k_j - k + 1}$,
 for certain $k_j \geq k$ and $A_j \in \Sigma_{k_j}$. Call each such a_j a *p -subatom*.

Given a martingale p -atomic block, set

$$|b|_{\text{atb},p}^1 = \begin{cases} \int_{\Omega} |b(\omega)| d\mu(\omega) & \text{when } b \in L_1(\Omega, \Sigma_1, \mu), \\ \inf_{\substack{b = \sum_j \lambda_j a_j \\ a_j \text{ } p\text{-subatom}}} \sum_{j \geq 1} |\lambda_j| & \text{when } b \notin L_1(\Omega, \Sigma_1, \mu). \end{cases}$$

Then we define the atomic block Hardy spaces

$$\begin{aligned} H_{\text{atb}}^1(\Omega) &= \left\{ f \in L_1(\Omega) \mid f = \sum_i b_i, \text{ } b_i \text{ martingale 2-atomic block} \right\}, \\ H_{\text{atb},p}^1(\Omega) &= \left\{ f \in L_1(\Omega) \mid f = \sum_i b_i, \text{ } b_i \text{ martingale } p\text{-atomic block} \right\}, \end{aligned}$$

which come equipped with the norm

$$\|f\|_{H_{\text{atb},p}^1(\Omega)} = \inf_{\substack{f = \sum_i b_i \\ b_i \text{ } p\text{-atomic block}}} \sum_{i \geq 1} |b_i|_{\text{atb},p}^1 = \inf_{\substack{f = \sum_i b_i \\ b_i = \sum_j \lambda_{ij} a_{ij}}} \sum_{i,j \geq 1} |\lambda_{ij}|,$$

where the a_{ij} 's above are taken to be p -subatoms of b_i . Note that $\lambda_{ij} = \delta_{j1} \|b_i\|_1$ for atomic blocks $b_i \in L_1(\Omega, \Sigma_1, \mu)$. With this definition of atomic blocks, $H_1 \rightarrow L_1$ boundedness reduces to

$$\|T(b)\|_1 \leq c_0 |b|_{\text{atb},p}^1$$

under mild regularity conditions for some c_0 independent of the p -atomic block b .

Theorem A. *There exists an isomorphism*

$$H_1(\Omega) \simeq H_{\text{atb},p}^1(\Omega) \quad \text{for } 1 < p \leq \infty.$$

In fact, an analogous result holds also for noncommutative martingales.

We have deliberately omitted the definition of atomic block for noncommutative martingales, which is postponed to Section 1. We shall only provide one proof of Theorem A which is valid for noncommutative martingales, although two additional arguments will be given in the commutative setting. Our main proof is perhaps the simplest one, relying on a noncommutative form of Davis decomposition from [13, 21]. An alternative proof exploits a weaker notion of atom which might be folklore or at least well-known to experts. It however requires to approximate general filtrations by atomic ones, something which seems to be out of the scope in the noncommutative setting. Our last proof avoids such approximation argument adapting Tolsa's argument [26] with conditional medians instead of medians. Our noncommutative results are in line with [1, 10, 21].

1. Noncommutative martingales

The theory of noncommutative martingale inequalities started with Cuculescu [5], but it did not receive significant attention until the work of Pisier/Xu [24] about the noncommutative analogue of Burkholder/Gundy inequalities. After it, most of the classical results on martingale L_p inequalities have found a noncommutative analogue, see [10, 12, 13, 14, 15, 16, 19, 20, 21, 29] and the references therein for basic definitions and results. Here we shall just introduce martingale p -atomic blocks and related notions in the noncommutative setting.

A noncommutative probability space is a pair (\mathcal{M}, τ) formed by a von Neumann algebra \mathcal{M} and a normal faithful finite trace τ , normalized so that $\tau(\mathbf{1}_{\mathcal{M}}) = 1$ for the unit $\mathbf{1}_{\mathcal{M}}$ of \mathcal{M} . A filtration in \mathcal{M} is an increasing sequence $(\mathcal{M}_k)_{k \geq 1}$ of von Neumann subalgebras of \mathcal{M} satisfying that their union is weak- $*$ dense in \mathcal{M} . Assume there exists a normal conditional expectation

$$E_k : \mathcal{M} \rightarrow \mathcal{M}_k$$

for every $k \geq 1$. Each E_k is trace preserving, unital and completely positive. In particular, $E_k : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}_k)$ defines a contraction for $1 \leq p \leq \infty$. These maps satisfy the bimodule property $E_k(\alpha f \beta) = \alpha E_k(f) \beta$ for $\alpha, \beta \in \mathcal{M}_k$. If we set $\Delta_k = E_k - E_{k-1}$ and write $E_k f = f_k$ and $\Delta_k f = df_k$ for $f \in L_1(\mathcal{M})$ (as in the commutative setting) then $H_1(\mathcal{M})$ is defined as the subspace of operators $f \in L_1(\mathcal{M})$ with finite norm

$$\|f\|_{H_1(\mathcal{M})} = \inf_{\substack{f=g+h \\ g, h \in L_1(\mathcal{M})}} \|g\|_{H_1^c(\mathcal{M})} + \|h^*\|_{H_1^c(\mathcal{M})},$$

where the column Hardy norm is given by

$$\|f\|_{H_1^c(\mathcal{M})} = \left\| \left(\sum_{k \geq 1} df_k^* df_k \right)^{\frac{1}{2}} \right\|_1.$$

The little Hardy space is defined similarly with

$$\|f\|_{h_1^c(\mathcal{M})} = \left\| \left(\sum_{k \geq 1} E_{k-1}(df_k^* df_k) \right)^{\frac{1}{2}} \right\|_1.$$

On the other hand, $BMO(\mathcal{M})$ is the subspace of $L_2(\mathcal{M})$ with

$$\|f\|_{BMO(\mathcal{M})} = \max \left\{ \|f\|_{BMO_c(\mathcal{M})}, \|f^*\|_{BMO_c(\mathcal{M})} \right\}$$

where the column BMO norm is given by the following expression

$$\|f\|_{BMO_c(\mathcal{M})} = \sup_{k \geq 1} \left\| \left(E_k((f - E_{k-1}f)^*(f - E_{k-1}f)) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}.$$

Of course, $bmo(\mathcal{M})$ arises when we replace E_{k-1} by E_k in the identity above.

We are now ready to define martingale p -atomic blocks in the noncommutative setting. As expected, we find row and column forms of these objects. We will say that an (unbounded) operator b affiliated with the von Neumann algebra \mathcal{M} is a *column martingale p -atomic block* when $b \in L_1(\mathcal{M}_1, \tau)$ or there exists an index $k \geq 1$ such that

- $E_k(b) = 0$,

- $b = \sum_j \lambda_j a_j$ where
 - $a_j q_j = a_j$,
 - $\|a_j\|_p \leq \tau(q_j)^{-\frac{1}{p'}} \frac{1}{k_j - k + 1}$,
 for some $k_j \geq k$ and projections $q_j \in \mathcal{M}_{k_j}$.

Each such a_j will be called a *column p -subatom*. Similarly, *row p -atomic blocks* are defined when the support identity $q_j a_j = a_j$ holds instead. In particular, both conditions hold for self-adjoint atomic blocks. Given a column p -atomic block b set

$$|b|_{\text{atb},p}^{1,c} = \tau(|b|)$$

when $b \in L_1(\mathcal{M}_1, \tau)$ and otherwise

$$|b|_{\text{atb},p}^{1,c} = \inf_{\substack{b = \sum_j \lambda_j a_j \\ a_j \text{ } p\text{-subatom}}} \sum_{j \geq 1} |\lambda_j|.$$

Then we define the atomic block Hardy spaces

$$\begin{aligned} H_{\text{atb}}^{1,c}(\mathcal{M}) &= \left\{ f \in L_1(\mathcal{M}) \mid f = \sum_i b_i, \text{ } b_i \text{ column 2-atomic block} \right\}, \\ H_{\text{atb},p}^{1,c}(\mathcal{M}) &= \left\{ f \in L_1(\mathcal{M}) \mid f = \sum_i b_i, \text{ } b_i \text{ column } p\text{-atomic block} \right\}, \end{aligned}$$

which come equipped with the following norm

$$\|f\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} = \inf_{\substack{f = \sum_i b_i \\ b_i \text{ } p\text{-atomic block}}} \sum_{i \geq 1} |b_i|_{\text{atb},p}^{1,c} = \inf_{\substack{f = \sum_i b_i \\ b_i = \sum_j \lambda_{ij} a_{ij}}} \sum_{i,j \geq 1} |\lambda_{ij}|,$$

where the a_{ij} 's above are taken to be p -subatoms of b_i . As in the commutative case, we pick $\lambda_{ij} = \delta_{j1} \|b_i\|_1$ for atomic blocks $b_i \in L_1(\mathcal{M}_1, \tau)$. Before stating the analogue of Theorem A for noncommutative martingales, we shall need the following approximation lemma to legitimate our duality argument below.

Lemma 1.1. *Given $\varepsilon > 0$ and*

$$f \in H_{\text{atb},p}^{1,c}(\mathcal{M}),$$

there exist a finite family $(b_i(\varepsilon))_{i \leq M}$ of column p -atomic blocks with

- i) $b_i(\varepsilon) \in L_p(\mathcal{M})$,
- ii) $\left\| f - \sum_{i=1}^M b_i(\varepsilon) \right\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} < \varepsilon$.
- iii) $\sum_{i=1}^M |b_i(\varepsilon)|_{\text{atb},p}^{1,c} < \left\| \sum_{i=1}^M b_i(\varepsilon) \right\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} + \varepsilon$.

Proof. Let $f = \sum_i b_i$ be such that

$$\begin{aligned} \left\| f - \sum_{i=1}^M b_i \right\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} &< \delta, \\ \left| \|f\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} - \sum_{i=1}^M |b_i|_{\text{atb},p}^{1,c} \right| &< \delta, \end{aligned}$$

with $\delta = \delta(\varepsilon)$ small and $M = M(\delta)$ large enough. From these properties it is clear that all the assertions in the statement will follow as long as we can show that every column p -atomic block b can be δ -approximated by another column p -atomic block b' living in $L_p(\mathcal{M})$. Indeed, when $b \in L_1(\mathcal{M}_1, \tau)$ it suffices to select an element $b' \in L_p(\mathcal{M}_1, \tau) \subset L_1(\mathcal{M}_1, \tau)$ with $\|b - b'\|_1 < \delta$. Otherwise

$$b = \sum_j \lambda_j a_j \quad \text{with} \quad E_k(b) = 0$$

is a sum of column p -subatoms. In that case, set $N = N(\delta)$ so that

$$\sum_{j > N} |\lambda_j| < \frac{\delta}{2k}$$

and define

$$b' = \sum_{j \leq N} \lambda_j a_j + E_1 \left(\sum_{j > N} \lambda_j a_j \right) =: \sum_{j \leq N+1} \lambda'_j a'_j.$$

According to the definition of column p -atomic block, the following holds

- $E_1(b') = E_1(b) = E_1 E_k(b) = 0$,
- If $(k'_j, q'_j) = (k_j, q_j)$ for $j \leq N$ and $(k'_{N+1}, q'_{N+1}) = (1, \mathbf{1}_{\mathcal{M}})$

$$a'_j q'_j = a'_j \quad \text{and} \quad \|a'_j\|_p \leq k \tau(q'_j)^{-\frac{1}{p'}} \frac{1}{k_j - 1 + 1}$$

provided we normalize a'_{N+1} so that $\lambda'_{N+1} = \|E_1(\sum_{j > N} \lambda_j a_j)\|_1$.

This shows that b' is a column p -atomic block. Moreover

$$\|b'\|_p = \left\| \sum_{j \leq N} \lambda_j a_j - E_1 \left(\sum_{j \leq N} \lambda_j a_j \right) \right\|_p \leq 2 \sum_{j \leq N} |\lambda_j| \|a_j\|_p < \infty.$$

Therefore, it just remains to prove the following estimate

$$\|b - b'\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} < \delta.$$

To that aim we identify $b - b'$ as a column p -atomic block

$$b - b' = \sum_{j > N} \lambda_j a_j - E_1 \left(\sum_{j > N} \lambda_j a_j \right) =: \sum_{j > N} \tilde{\lambda}_j \tilde{a}_j + \tilde{\lambda}_N \tilde{a}_N$$

with \tilde{a}_N normalized so that $\tilde{\lambda}_N = \|E_1(\sum_{j > N} \lambda_j a_j)\|_1$. Then we find

- $E_1(b - b') = 0$,
- If $(\tilde{k}_j, \tilde{q}_j) = (k_j, q_j)$ for $j > N$ and $(\tilde{k}_N, \tilde{q}_N) = (1, \mathbf{1}_{\mathcal{M}})$

$$\tilde{a}_j \tilde{q}_j = \tilde{a}_j \quad \text{and} \quad \|\tilde{a}_j\|_p \leq k \tau(\tilde{q}_j)^{-\frac{1}{p'}} \frac{1}{k_j - 1 + 1}.$$

This makes it quite simple to estimate the $H_{\text{atb},p}^{1,c}(\mathcal{M})$ -norm of $b - b'$

$$\begin{aligned} \|b - b'\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} &\leq \|b - b'\|_{\text{atb},p}^{1,c} \\ &\leq k \left[\sum_{j > N} |\lambda_j| + \left\| E_1 \left(\sum_{j > N} \lambda_j a_j \right) \right\|_1 \right] \\ &\leq k \left[\sum_{j > N} |\lambda_j| + \sum_{j > N} |\lambda_j| \|a_j\|_1 \right] \leq 2k \sum_{j > N} |\lambda_j| < \delta. \end{aligned}$$

Here we used the inequality $\|a_j\|_1 = \|a_j q_j\|_1 \leq \|a_j\|_p \tau(q_j)^{\frac{1}{p'}} \leq \frac{1}{k_j - k + 1} \leq 1$. \square

Theorem 1.2. *There exists an isomorphism*

$$H_1^c(\mathcal{M}) \simeq H_{\text{atb},p}^{1,c}(\mathcal{M}) \quad \text{for } 1 < p \leq \infty.$$

In particular, we find the atomic block decomposition $H_1(\mathcal{M}) \simeq H_{\text{atb},p}^1(\mathcal{M})$.

Proof. We need to show

- i) $H_{\text{atb},p}^{1,c}(\mathcal{M}) \subset H_1^c(\mathcal{M})$,
- ii) $H_1^c(\mathcal{M}) \subset H_{\text{atb},p}^{1,c}(\mathcal{M})$.

Step 1. For the first continuous inclusion we shall prove

$$\text{BMO}_c(\mathcal{M}) \subset H_{\text{atb},p}^{1,c}(\mathcal{M})^*,$$

which suffices by duality. Assume that $\phi \in \text{BMO}_c(\mathcal{M})$. Since $\phi \in L_{p'}(\mathcal{M})$ for any $1 < p \leq \infty$, we may represent ϕ as a linear functional L_ϕ on $L_p(\mathcal{M})$ by the formula

$$L_\phi(f) = \tau(f\phi^*).$$

According to Lemma 1.1, it suffices to show that

$$|L_\phi(f)| \leq \|f\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} \|\phi\|_{\text{BMO}_c(\mathcal{M})}$$

for every f which can be written as a finite sum $f = \sum_i b_i$ of column p -atomic blocks $b_i \in L_p(\mathcal{M})$. This clearly allows us to estimate $|L_\phi(f)| \leq \sum_i |L_\phi(b_i)|$ with the right hand side well-defined. In particular, it is enough to show that

$$|L_\phi(b)| \lesssim \|b\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} \|\phi\|_{\text{BMO}_c(\mathcal{M})}$$

for column p -atomic blocks $b \in L_p(\mathcal{M})$. When $b \in L_p(\mathcal{M}_1, \tau)$

$$|L_\phi(b)| \leq \|b\|_1 \|\mathbf{E}_1 \phi\|_\infty \leq \|b\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} \|\phi\|_{\text{BMO}_c(\mathcal{M})}.$$

Otherwise, we write $b = \sum_j \lambda_j a_j$ with $\mathbf{E}_k(b) = 0$ and such that

$$a_j q_j = a_j \quad , \quad \|a_j\|_p \leq \tau(q_j)^{-\frac{1}{p'}} \frac{1}{k_j - k + 1}$$

for some $k_j \geq k$ and some projection $q_j \in \mathcal{M}_{k_j}$. Then we find that

$$|L_\phi(b)| = |\tau(b\phi^*)| = |\tau(b(\phi - \mathbf{E}_k \phi)^*)| \leq \sum_j |\lambda_j| \|a_j(\phi - \mathbf{E}_k \phi)^*\|_1 =: \sum_j |\lambda_j| A_j.$$

Hence, it remains to prove that $\sup_j A_j \lesssim \|\phi\|_{\text{BMO}_c(\mathcal{M})}$, which follows from

$$\begin{aligned} A_j &\leq \|a_j\|_p \|(\phi - \mathbf{E}_k \phi) q_j\|_{p'} \\ &\leq \tau(q_j)^{-\frac{1}{p'}} \frac{1}{k_j - k + 1} \|(\phi - \mathbf{E}_k \phi) q_j\|_{p'} \\ &\leq \tau(q_j)^{-\frac{1}{p'}} \|(\phi - \mathbf{E}_{k_j} \phi) q_j\|_{p'} + \frac{1}{k_j - k + 1} \sum_{s=k+1}^{k_j} \|d\phi_s\|_\infty = B_j + C_j \end{aligned}$$

Indeed, this yields the estimate

$$B_j + C_j \leq \|\phi\|_{\text{bmo}_c(\mathcal{M})} + \sup_{k \geq 1} \|d\phi_k\|_\infty \simeq \|\phi\|_{\text{BMO}_c(\mathcal{M})}$$

where the inequality $B_j \leq \|\phi\|_{\text{bmo}_c(\mathcal{M})}$ follows from Hong/Mei formulation of the John-Nirenberg inequality for noncommutative martingales [10]. In particular, this completes the proof of Step 1.

Step 2. We now prove the inclusion

$$H_1^c(\mathcal{M}) \subset H_{\text{atb},p}^{1,c}(\mathcal{M})$$

directly, without using duality. Here we would like to thank Marius Junge for suggesting us the noncommutative Davis decomposition (used below) as a possible tool in proving this inclusion. Let $f \in H_1^c(\mathcal{M})$, by the noncommutative form of Davis decomposition [21] we know that f can be decomposed as $f = f_c + f_d$, where

$$(f_c, f_d) \in h_{\text{at},c}^1(\mathcal{M}) \times h_{\text{diag}}^1(\mathcal{M}).$$

On the other hand, since a column p -atom in the sense of [1, 10] is in particular a column p -atomic block in our sense, we immediately find the following inequality

$$\|f_c\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} \lesssim \|f\|_{H_1^c(\mathcal{M})}.$$

The diagonal norm of f_d is given by

$$\|f_d\|_{h_{\text{diag}}^1(\mathcal{M})} = \sum_{k \geq 1} \|\Delta_k(f_d)\|_1 \lesssim \|f\|_{H_1^c(\mathcal{M})}.$$

Therefore, the goal is to show that we have

$$\|f_d\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} \lesssim \|f_d\|_{h_{\text{diag}}^1(\mathcal{M})}.$$

Since the norm in $h_{\text{diag}}^1(\mathcal{M})$ is $*$ -invariant, we may assume that f_d is a self-adjoint operator. Then, by an L_p -approximation argument we may also assume that the martingale differences have the form

$$\Delta_k(f_d) = \sum_{j \geq 1} \beta_{jk} p_{jk} = \sum_{j \geq 1} \beta_{jk} \Delta_k(p_{jk})$$

for certain $\beta_{jk} \in \mathbb{R}$ and a family $(p_{jk})_{j \geq 1}$ of pairwise disjoint projections. We claim

$$|\Delta_k(p)|_{\text{atb},p}^{1,c} \lesssim \tau(p)$$

for any projection p . This is enough to conclude since then

$$\begin{aligned} \|f_d\|_{H_{\text{atb},p}^{1,c}(\mathcal{M})} &\leq \sum_{j,k \geq 1} |\beta_{jk}| |\Delta_k(p_{jk})|_{\text{atb},p}^{1,c} \\ &\lesssim \sum_{j,k \geq 1} |\beta_{jk}| \tau(p_{jk}) = \sum_{k \geq 1} \left\| \sum_{j \geq 1} \beta_{jk} p_{jk} \right\|_1 = \|f_d\|_{h_{\text{diag}}^1(\mathcal{M})}. \end{aligned}$$

Let us then prove our claim for $b = \Delta_k(p)$. To show that b is a column p -atomic block, we start by noticing $E_{k-1}(b) = 0$. Let us introduce the family of projections

$$\begin{aligned} q_j(k) &= \chi_{(\frac{1}{j+1}, \frac{1}{j}]}(E_k p), \\ q_j(k-1) &= \chi_{(\frac{1}{j+1}, \frac{1}{j}]}(E_{k-1} p). \end{aligned}$$

Decompose b into column p -subatoms as follows

$$b = \sum_{j \geq 1} \lambda_j(k) a_j(k) - \lambda_j(k-1) a_j(k-1)$$

where coefficients and subatoms are respectively given by

$$\begin{aligned} \lambda_j(k) &= \frac{2}{j} \tau(q_j(k)), \\ \lambda_j(k-1) &= \frac{1}{j} \tau(q_j(k-1)), \\ a_j(k) &= \lambda_j(k)^{-1} q_j(k) E_k(p), \end{aligned}$$

$$a_j(k-1) = \lambda_j(k-1)^{-1} q_j(k-1) E_{k-1}(p),$$

Since $(q_j(k-1), q_j(k)) \in \mathcal{M}_{k-1} \times \mathcal{M}_k$, we will have a column p -atomic block b if

- $a_j(k-1)q_j(k-1) = a_j(k-1)$ and $a_j(k)q_j(k) = a_j(k)$,
- $\|a_j(k-1)\|_p \leq \tau(q_j(k-1))^{-\frac{1}{p'}}$ and $\|a_j(k)\|_p \leq \frac{1}{2}\tau(q_j(k))^{-\frac{1}{p'}}$.

It is however a simple exercise to check that this is indeed the case and

$$\begin{aligned} |b|_{\text{atb},p}^{1,c} &\leq \sum_{j \geq 1} |\lambda_j(k)| + |\lambda_j(k-1)| \\ &\leq \sum_{j \geq 1} 4\tau(q_j(k)E_k(p)) + 2\tau(q_j(k-1)E_{k-1}(p)) \leq 6\tau(p). \end{aligned}$$

This justifies our claim above and hence completes the proof of the assertion. \square

Remark 1.3. The noncommutative Davis decomposition of Perrin and Junge/Mei [13, 21] is sometimes referred to as the “atomic decomposition” for $H_1^c(\mathcal{M})$, since it relates this space with the atomic Hardy space $h_{\text{at},c}^1(\mathcal{M})$ and the diagonal space $h_{\text{diag}}^1(\mathcal{M})$. Nevertheless, it seems there is no atomic decomposition of the diagonal part (in the noncommutative setting) beyond the results in this paper.

2. Two alternative arguments for classical martingales

In this section we explore two additional proofs of Theorem A valid for classical martingales. None of them work for noncommutative martingales, but shed some light to the problem. The first one uses a weaker notion of atom which proves that atomic blocks can be taken with (at most) two subatoms. Notice that this does not seem to be the case in the von Neumann algebra setting. This is analogous to a similar result for Tolsa’s atomic blocks. The second one illustrates how conditional medians instead of medians allow to give a direct proof, avoiding approximation by atomic filtrations. Moreover, we shall obtain in the process an equivalent expression $\|f\|_{\text{BMO}}^\alpha$ for the martingale BMO norm of f .

2.1. A proof using weak atoms. Given a probability space (Ω, Σ, μ) and any filtration $(\Sigma_k)_{k \geq 1}$, we will say that a measurable function $w : \Omega \rightarrow \mathbb{C}$ is a *weak ∞ -atom* when there is some $k \geq 1$, a Σ_k -measurable function $\varphi : \Omega \rightarrow \mathbb{C}$, with $|\varphi| \leq 1$ and $A := \text{supp } \varphi \in \Sigma_k$ so that

$$w = \frac{\varphi - E_{k-1}(\varphi)}{\mu(A)}.$$

We may find such kind of atoms in [2], but perhaps they were known before. Let us sketch the proof of Theorem A for $p = \infty$ using weak ∞ -atoms. The proof of the inclusion $H_{\text{atb},\infty}^1(\Omega) \subset H_1(\Omega)$ will not change from our first proof of Theorem A above. By a straightforward approximation argument (that we will not reproduce here) we may assume that our filtration $(\Sigma_k)_{k \geq 1}$ is atomic. Under this assumption all we need to prove by duality is that

$$\|f\|_{\text{BMO}(\Omega)} \lesssim \sup_{\substack{b \text{ atomic block} \\ |b|_{\text{atb},\infty}^1 \leq 1}} \int_{\Omega} f b \, d\mu$$

holds for any $f \in \text{BMO}(\Omega)$. Let us briefly justify this, consider $f \in \text{BMO}(\Omega)$. Since we assume $(\Sigma_k)_{k \geq 1}$ is atomic, given any $\varepsilon > 0$ we may find certain $k \geq 1$ and an atom $A \in \Sigma_k$ such that

$$\|f\|_{\text{BMO}(\Omega)} < (1 + \varepsilon) \mathbb{E}_k |f - \mathbb{E}_{k-1}(f)| (A).$$

On the other hand, we have

$$\begin{aligned} & \mathbb{E}_k |f - \mathbb{E}_{k-1}(f)| (A) \\ &= \sup_{\|\xi\|_\infty \leq 1} \frac{1}{\mu(A)} \int_A \xi(f - \mathbb{E}_{k-1}(f)) d\mu \\ &= \sup_{\|\xi\|_\infty \leq 1} \frac{1}{\mu(A)} \int_A \xi(f - \mathbb{E}_k(f) + \mathbb{E}_k(f) - \mathbb{E}_{k-1}(f)) d\mu \\ &= \sup_{\|\xi\|_\infty \leq 1} \int_\Omega \underbrace{\chi_A \frac{\xi - \mathbb{E}_k(\xi)}{\mu(A)}}_{a(\xi)} f d\mu + \int_\Omega \underbrace{\frac{\mathbb{E}_k(\chi_A \xi) - \mathbb{E}_{k-1}(\chi_A \xi)}{\mu(A)}}_{w(\xi)} f d\mu. \end{aligned}$$

It is clear that $a(\xi)$ is an ∞ -atom with

$$|a(\xi)|_{\text{atb}, \infty}^1 \leq 2.$$

Therefore, it suffices to show that $w(\xi)$ is an ∞ -atomic block with

$$|w(\xi)|_{\text{atb}, \infty}^1 \lesssim 1.$$

Note that $w(\xi)$ is a weak ∞ -atom which can be written as $a_1(\xi) + a_2(\xi)$ with

$$\begin{aligned} a_1(\xi) &= \frac{-1}{\mu(A)} \left(\frac{1}{\mu(B)} \int_\Omega \chi_A \xi d\mu \right) \chi_{B \setminus A}, \\ a_2(\xi) &= \frac{1}{\mu(A)} \left(\mathbb{E}_k(\xi) - \frac{1}{\mu(B)} \int_\Omega \chi_A \xi d\mu \right) \chi_A, \end{aligned}$$

where B is the only atom in Σ_{k-1} containing the atom $A \in \Sigma_k$. Now, it all reduces to show that (up to absolute constants) $a_1(\xi)$ and $a_2(\xi)$ are ∞ -subatoms. Using that $|\xi| \leq 1$, we deduce

$$|a_1(\xi)| \leq \frac{\chi_{B \setminus A}}{\mu(B)} \leq \frac{\chi_{B \setminus A}}{\mu(B \setminus A)} \quad \text{and} \quad |a_2(\xi)| \leq \frac{2\chi_A}{\mu(A)}.$$

Since $a_1(\xi), a_2(\xi)$ are Σ_k -measurable, we easily get the estimate $|w(\xi)|_{\text{atb}, \infty}^1 \leq 6$.

This argument shows that atomic blocks in Theorem A can be taken with at most two subatoms for classical martingales. Unfortunately, the argument does not extend to the noncommutative setting, because of the lack of an approximation argument by atomic filtrations.

2.2. A proof using conditional medians. Given a probability space (Ω, Σ, μ) and a σ -subalgebra $\Sigma_0 \subset \Sigma$, a conditional median $\alpha_0 f$ of a Σ -measurable function f is a random variable which satisfies:

- $\alpha_0 f$ is Σ_0 -measurable,
- Given any $A \in \Sigma_0$, we have

$$\max \left\{ \mu(A \cap \{f > \alpha_0 f\}), \mu(A \cap \{f < \alpha_0 f\}) \right\} \leq \frac{1}{2} \mu(A).$$

Tomkins theorem [27] shows that each random variable has at least one conditional median with respect to any given σ -algebra. In the sequel, we will denote a fixed conditional median of f with respect to Σ_k by $\alpha_k f$. Before the proof of Theorem A we need a simple lemma which will be crucial in our argument.

Lemma 2.1. *Given $A \in \Sigma_0$ and f Σ -measurable*

$$E_0(\chi_{A \cap \{f \leq \alpha_0 f\}}) \geq \frac{1}{2} \chi_A \quad \mu\text{-a.e.}$$

where E_0 denotes the conditional expectation onto the σ -subalgebra $\Sigma_0 \subset \Sigma$.

Proof. By the definition of conditional median

$$\mu(B \cap \{f \leq \alpha_0 f\}) \geq \frac{1}{2} \mu(B)$$

for every Σ_0 -measurable set B . Assume now that the set A in the statement fails the given inequality and define B to be the Σ_0 -measurable level set where $E_0(\chi_{A \cap \{f \leq \alpha_0 f\}}) < \frac{1}{2}$. If the assertion failed for A , we would have $\mu(B) > 0$ and we could conclude that

$$\begin{aligned} \mu(B \cap \{f \leq \alpha_0 f\}) &= \int_B E_0(\chi_{B \cap \{f \leq \alpha_0 f\}}) d\mu \\ &\leq \int_B E_0(\chi_{A \cap \{f \leq \alpha_0 f\}}) d\mu < \frac{1}{2} \mu(B) \end{aligned}$$

which contradicts the definition of conditional median. The proof is complete. \square

Proof of Theorem A for $p < \infty$. Again, the proof of $H_{\text{atb},p}^1(\Omega) \subset H_1(\Omega)$ will not change from our first proof of Theorem A above. Here we only prove the reverse inclusion $H_1(\Omega) \subset H_{\text{atb},p}^1(\Omega)$. To that end, we will show that

$$H_{\text{atb},p}^1(\Omega)^* \subset \text{BMO}(\Omega),$$

which suffices by duality. Let $L : H_{\text{atb},p}^1(\Omega) \rightarrow \mathbb{C}$ be a continuous functional in the dual space. To proceed, we need to show that $L = L_f$ acts by integration in (Ω, μ) against a function $f \in L_{\text{loc}}^1(\Omega)$ and deduce a posteriori that $f \in \text{BMO}(\Omega)$ and we have

$$\|f\|_{\text{BMO}(\Omega)} \leq C_p \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*}$$

for some absolute constant C_p . The existence of such f follows from the inclusion $h_{\text{at},p}^1(\Omega) \subset H_{\text{atb},p}^1(\Omega)$, so that $H_{\text{atb},p}^1(\Omega)^* \subset h_{\text{at},p}^1(\Omega)^* = \text{bmo}(\Omega)$. In particular any continuous functional L in the dual of $H_{\text{atb},p}^1(\Omega)$ can be represented by a function $f \in \text{bmo}(\Omega)$. We now claim that

$$\frac{1}{c_p} \|f\|_{\text{BMO}(\Omega)} \leq \|f\|_{\text{BMO}}^\alpha \leq c_p \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*},$$

where c_p only depends on p and $\|f\|_{\text{BMO}}^\alpha$ is given by

$$\|f\|_{\text{BMO}}^\alpha = \max \left\{ \|E_1 f\|_\infty, \sup_{k \geq 1} \|E_k |f - \alpha_k f|^{p'}\|_\infty^{\frac{1}{p'}}, \sup_{k \geq 2} \|\alpha_k f - \alpha_{k-1} f\|_\infty \right\}.$$

Note that this quantity depends a priori on the choice of the conditional medians $\alpha_k f$. This however will be unsubstantial since our inequalities hold with constants which are independent of our choice. It is clear that the proof will be complete if we justify our claim, which we will in two steps.

Step 1. The inequality

$$\|f\|_{\text{BMO}(\Omega)} \leq c_p \|f\|_{\text{BMO}^\alpha}$$

is the simplest one. Namely, by John-Nirenberg inequality we have

$$\begin{aligned} \|f\|_{\text{BMO}(\Omega)} &= \sup_{k \geq 1} \left\| \mathbb{E}_k |f - \mathbb{E}_{k-1} f|^2 \right\|_\infty^{\frac{1}{2}} \\ &\sim \|\mathbb{E}_1 f\|_\infty + \sup_{k \geq 1} \left\| \mathbb{E}_k |f - \mathbb{E}_k f|^2 \right\|_\infty^{\frac{1}{2}} + \sup_{k \geq 2} \|df_k\|_\infty \\ &\sim \|\mathbb{E}_1 f\|_\infty + \sup_{k \geq 1} \left\| \mathbb{E}_k |f - \mathbb{E}_k f|^{p'} \right\|_\infty^{\frac{1}{p'}} + \sup_{k \geq 2} \|df_k\|_\infty = A_1 + A_2 + A_3. \end{aligned}$$

The term A_1 admits a trivial bound. Next

$$\begin{aligned} A_2 &\leq \sup_{k \geq 1} \left\| \mathbb{E}_k |f - \alpha_k f|^{p'} \right\|_\infty^{\frac{1}{p'}} + \left\| \mathbb{E}_k |\alpha_k f - \mathbb{E}_k f|^{p'} \right\|_\infty^{\frac{1}{p'}} \\ &\leq \|f\|_{\text{BMO}}^\alpha + \sup_{k \geq 1} \left\| \mathbb{E}_k (f - \alpha_k f) \right\|_\infty \leq 2 \|f\|_{\text{BMO}}^\alpha, \end{aligned}$$

where the last inequality uses conditional Jensen's inequality $\phi(\mathbb{E}_k f) \leq \mathbb{E}_k(\phi(f))$ for the convex function $\phi(x) = x^{p'}$. Finally, the last term A_3 is estimated by decomposing $df_k = \mathbb{E}_k(f - \alpha_k f) + (\alpha_k f - \alpha_{k-1} f) - \mathbb{E}_{k-1}(f - \alpha_{k-1} f)$ together with the triangle inequality and conditional Jensen's inequality one more time.

Step 2. The inequality

$$\|f\|_{\text{BMO}}^\alpha \leq c_p \|L_f\|_{\text{H}_{\text{atb},p}^1(\Omega)^*}$$

requires a bit more work. Since Σ_1 -measurable functions are atomic blocks

$$\|\mathbb{E}_1 f\|_\infty = \sup_{B \in \Sigma_1} \left| \int_B f d\mu \right| \leq \frac{1}{\mu(B)} \|L_f\|_{\text{H}_{\text{atb},p}^1(\Omega)^*} \|\chi_B\|_{\text{H}_{\text{atb},p}^1(\Omega)} \leq \|L_f\|_{\text{H}_{\text{atb},p}^1(\Omega)^*}.$$

Let us now bound the other two terms in $\|f\|_{\text{BMO}}^\alpha$. In order to estimate the second term, we will use that for any $A \in \Sigma_k$ there exists a p -atomic block $b_{A,f} \neq 0$ satisfying the following two inequalities

$$\|b_{A,f}\|_{\text{H}_{\text{atb},p}^1(\Omega)} \lesssim \mu(A)^{\frac{1}{p'}} \left(\int_A |f - \alpha_k f|^{p'} d\mu \right)^{\frac{1}{p}} \lesssim \mu(A)^{\frac{1}{p'}} \left| \int_\Omega f b_{A,f} d\mu \right|^{\frac{1}{p}}.$$

This immediately implies that

$$\sup_{k \geq 1} \left\| \mathbb{E}_k |f - \alpha_k f|^{p'} \right\|_\infty^{\frac{1}{p'}} \lesssim \|L_f\|_{\text{H}_{\text{atb},p}^1(\Omega)^*}$$

as desired. Indeed, this can be justified as follows

$$\begin{aligned} \left\| \mathbb{E}_k |f - \alpha_k f|^{p'} \right\|_\infty^{\frac{1}{p'}} &= \sup_{A \in \Sigma_k} \left(\int_A |f - \alpha_k f|^{p'} d\mu \right)^{\frac{1}{p'}} \\ &\lesssim \sup_{A \in \Sigma_k} \frac{1}{\|b_{A,f}\|_{\text{H}_{\text{atb},p}^1(\Omega)}} \left| \int_\Omega f b_{A,f} d\mu \right| \leq \|L_f\|_{\text{H}_{\text{atb},p}^1(\Omega)^*}. \end{aligned}$$

Given $A \in \Sigma_k$, let us then prove the existence of such p -atomic block. Assume

$$\int_{A \cap \{f > \alpha_k f\}} |f - \alpha_k f|^{p'} d\mu \geq \int_{A \cap \{f < \alpha_k f\}} |f - \alpha_k f|^{p'} d\mu.$$

This assumption is admissible since we may easily modify the construction of our p -atomic block $b_{A,f}$ to satisfy the required estimates in case the inequality above is reversed. Define the function

$$b_{A,f}(x) = |f - \alpha_k f|^{p'-1} \chi_{A \cap \{f > \alpha_k f\}} - \frac{\mathbb{E}_k(|f - \alpha_k f|^{p'-1} \chi_{A \cap \{f > \alpha_k f\}})}{\mathbb{E}_k(\chi_{A \cap \{f \leq \alpha_k f\}})} \chi_{A \cap \{f \leq \alpha_k f\}}.$$

Obviously, $\mathbb{E}_k(b_{A,f}) = 0$ and $\text{supp}(b_{A,f}) \subset A$. This yields

$$\begin{aligned} \|b_{A,f}\|_{H_{\text{atb},p}^1(\Omega)} &\leq \mu(A)^{\frac{1}{p'}} \|b_{A,f}\|_p \\ &\leq \mu(A)^{\frac{1}{p'}} \left(\int_{A \cap \{f > \alpha_k f\}} |f - \alpha_k f|^{p(p'-1)} d\mu \right)^{\frac{1}{p}} \\ &\quad + \mu(A)^{\frac{1}{p'}} \left(\int_{A \cap \{f \leq \alpha_k f\}} \left[\frac{\mathbb{E}_k(|f - \alpha_k f|^{p'-1})}{\mathbb{E}_k(\chi_{A \cap \{f \leq \alpha_k f\}})} \right]^p d\mu \right)^{\frac{1}{p}} = A_1 + A_2. \end{aligned}$$

Since $p(p' - 1) = p'$, A_1 clearly satisfies the desired estimate. On the other hand

$$\begin{aligned} A_2 &= \mu(A)^{\frac{1}{p'}} \left(\int_A \chi_{A \cap \{f \leq \alpha_k f\}} \left[\frac{\mathbb{E}_k(|f - \alpha_k f|^{p'-1})}{\mathbb{E}_k(\chi_{A \cap \{f \leq \alpha_k f\}})} \right]^p d\mu \right)^{\frac{1}{p}} \\ &= \mu(A)^{\frac{1}{p'}} \left(\int_A [\mathbb{E}_k(|f - \alpha_k f|^{p'-1})]^p [\mathbb{E}_k(\chi_{A \cap \{f \leq \alpha_k f\}})]^{1-p} d\mu \right)^{\frac{1}{p}} \\ &\lesssim \mu(A)^{\frac{1}{p'}} \left(\int_A [\mathbb{E}_k(|f - \alpha_k f|^{p'-1})]^p d\mu \right)^{\frac{1}{p}} \leq \mu(A)^{\frac{1}{p'}} \left(\int_A \mathbb{E}_k(|f - \alpha_k f|^{p'}) d\mu \right)^{\frac{1}{p}}, \end{aligned}$$

where we have used Lemma 2.1 for the first inequality and conditional Jensen's inequality for the second one. Now, since $A \in \Sigma_k$, we can remove the conditional expectation \mathbb{E}_k in the integrand of the last term above to complete the proof of the estimate for $\|b_{A,f}\|_{\text{atb},p}^1$. The other inequality is simpler. Since $(f - \alpha_k f)b_{A,f}$ is nonnegative by definition of $b_{A,f}$ and $\mathbb{E}_k(b_{A,f}) = 0$, we get

$$\begin{aligned} \int_{\Omega} f b_{A,f} d\mu &= \int_{\Omega} (f - \alpha_k f) b_{A,f} d\mu \\ &\geq \int_{A \cap \{f > \alpha_k f\}} |f - \alpha_k f|^{p'} d\mu \geq \frac{1}{2} \int_A |f - \alpha_k f|^{p'} d\mu. \end{aligned}$$

This completes the proof of the expected estimate for the second term in $\|f\|_{\text{BMO}}^\alpha$. It remains to prove that

$$\sup_{k \geq 2} \|\alpha_k f - \alpha_{k-1} f\|_\infty = \sup_{k \geq 2} \sup_{A \in \Sigma_k} \int_A |\alpha_k f - \alpha_{k-1} f| d\mu \lesssim c_p \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*}.$$

Fix $k > 1$ and $A \in \Sigma_k$. By the triangle and Jensen's inequality

$$\int_A |\alpha_k f - \alpha_{k-1} f| d\mu \leq \left(\int_A |f - \alpha_k f|^{p'} d\mu \right)^{\frac{1}{p'}} + \left(\int_A |f - \alpha_{k-1} f|^{p'} d\mu \right)^{\frac{1}{p'}}.$$

Since $A \in \Sigma_k$, the first term in the right hand side is bounded above by

$$\|\mathbb{E}_k |f - \alpha_k f|^{p'}\|_\infty^{\frac{1}{p'}} \lesssim \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*}$$

as we proved before. To bound the second term, we consider the function

$$b_{A,f} = \underbrace{\frac{|f - \alpha_{k-1}f|^{p'}}{f - \alpha_{k-1}f} \chi_{A \cap \{f \neq \alpha_{k-1}f\}}}_{\lambda_* a_*} - \underbrace{E_{k-1} \left(\frac{|f - \alpha_{k-1}f|^{p'}}{f - \alpha_{k-1}f} \chi_{A \cap \{f \neq \alpha_{k-1}f\}} \right)}_{\sum_{j \in \mathbb{Z}} \lambda_j a_j}$$

where

$$\lambda_j a_j = E_{k-1} \left(\frac{|f - \alpha_{k-1}f|^{p'}}{f - \alpha_{k-1}f} \chi_{A \cap \{f \neq \alpha_{k-1}f\}} \right) \underbrace{\chi_{\{2^{j-1} < E_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) \leq 2^j\}}}_{\chi_{B_j}}.$$

We have $E_{k-1}(b_{A,f}) = 0$ so that

$$\begin{aligned} \|b_{A,f}\|_{H_{\text{atb},p}^1(\Omega)} &\leq |\lambda_*| + \sum_{j \in \mathbb{Z}} |\lambda_j| \\ &\leq \mu(A)^{\frac{1}{p'}} \| |f - \alpha_{k-1}f|^{p'-1} \chi_A \|_p \\ &\quad + \sum_{j \in \mathbb{Z}} \mu(B_j)^{\frac{1}{p'}} \| E_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) \chi_{B_j} \|_p \end{aligned}$$

The second term in the right hand side is dominated by the first one since

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \mu(B_j)^{\frac{1}{p'}} \| E_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) \chi_{B_j} \|_p \\ &\leq \sum_{j \in \mathbb{Z}} 2^j \mu(B_j) \\ &\sim \sum_{j \in \mathbb{Z}} \int_{B_j} E_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) d\mu \mu(B_j) \\ &= \int_{\cup B_j} E_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) d\mu \\ &= \int_A |f - \alpha_{k-1}f|^{p'-1} d\mu \leq \mu(A)^{\frac{1}{p'}} \| |f - \alpha_{k-1}f|^{p'-1} \chi_A \|_p. \end{aligned}$$

In summary, we have proved that

$$\|b_{A,f}\|_{H_{\text{atb},p}^1(\Omega)} \lesssim \mu(A)^{\frac{1}{p'}} \| |f - \alpha_{k-1}f|^{p'-1} \chi_A \|_p.$$

On the other hand, let us observe that

$$\begin{aligned} \int_{\Omega} f b_{A,f} d\mu &= \int_{\Omega} (f - \alpha_{k-1}f) b_{A,f} d\mu \\ &= \int_A |f - \alpha_{k-1}f|^{p'} d\mu \\ &\quad - \int_{\Omega} (f - \alpha_{k-1}f) E_{k-1} \left(\frac{|f - \alpha_{k-1}f|^{p'}}{f - \alpha_{k-1}f} \chi_{A \cap \{f \neq \alpha_{k-1}f\}} \right) d\mu. \end{aligned}$$

Using this and the estimates so far we obtain

$$\int_A |f - \alpha_{k-1}f|^{p'} d\mu \leq \left| \int_{\Omega} f b_{A,f} d\mu \right|$$

$$\begin{aligned}
& + \sum_j \int_{B_j} |f - \alpha_{k-1}f| \mathbb{E}_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) d\mu \\
& \leq \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*} \|b_{A,f}\|_{H_{\text{atb},p}^1(\Omega)} \\
& + \sum_j \|\mathbb{E}_{k-1}|f - \alpha_{k-1}f|^{p'}\|_{\infty}^{\frac{1}{p'}} \mu(B_j)^{\frac{1}{p'}} \|\mathbb{E}_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) \chi_{B_j}\|_p \\
& \lesssim \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*} \|b_{A,f}\|_{H_{\text{atb},p}^1(\Omega)} \\
& + \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*} \sum_j \mu(B_j)^{\frac{1}{p'}} \|\mathbb{E}_{k-1}(|f - \alpha_{k-1}f|^{p'-1} \chi_A) \chi_{B_j}\|_p \\
& \lesssim \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*} \mu(A)^{\frac{1}{p'}} \| |f - \alpha_{k-1}f|^{p'-1} \chi_A \|_p.
\end{aligned}$$

Rearranging and noticing that $p(p' - 1) = p'$ we get

$$\left(\int_A |f - \alpha_{k-1}f|^{p'} d\mu \right)^{\frac{1}{p'}} \lesssim \|L_f\|_{H_{\text{atb},p}^1(\Omega)^*},$$

the desired estimate. This completes the proof of Theorem A for $p < \infty$. \square

Proof of Theorem A for $p = \infty$. The proof presents a lot of similarities with the case $p < \infty$. As above, we will only prove the inclusion $H_1(\Omega) \subset H_{\text{atb},\infty}^1(\Omega)$. Again, we proceed by duality and the goal is to show that

$$\|f\|_{\text{BMO}(\Omega)} \lesssim \|f\|_{\text{BMO}}^\alpha \lesssim \|L_f\|_{H_{\text{atb},\infty}^1(\Omega)^*}.$$

Our former argument for the first inequality is still valid. Now consider

- (1) There exists $k \geq 1$ and $A \in \Sigma_k$ such that

$$\int_A |f - \alpha_k f| d\mu \geq \frac{1}{32} \|f\|_{\text{BMO}}^\alpha.$$

- (2) Property (1) fails and there exists $k \geq 2$ such that

$$\|\alpha_k f - \alpha_{k-1} f\|_\infty \geq \frac{1}{2} \|f\|_{\text{BMO}}^\alpha.$$

- (3) The following inequality holds

$$\max \left\{ \sup_{k \geq 1} \|\mathbb{E}_k |f - \alpha_k f|^{p'}\|_{\infty}^{\frac{1}{p'}}, \sup_{k \geq 2} \|\alpha_k f - \alpha_{k-1} f\|_\infty \right\} \leq \|\mathbb{E}_1 f\|_\infty.$$

It is not difficult to check that at least one of the properties above always hold for every f with $\|f\|_{\text{BMO}}^\alpha$ finite. When (3) holds, we may argue as in the proof of the case $p < \infty$ to deduce $\|f\|_{\text{BMO}}^\alpha \leq \|L_f\|_{(H_{\text{atb},\infty}^1(\Omega))^*}$. When (1) holds, we consider the following function

$$b_{A,f} = \underbrace{\chi_{A \cap \{f > \alpha_k f\}} - \chi_{A \cap \{f < \alpha_k f\}}}_{b_1} - \underbrace{\chi_{A \cap \{f = \alpha_k f\}} \mathbb{E}_k(b_1) [\mathbb{E}_k(\chi_{A \cap \{f = \alpha_k f\}})]^{-1}}_{b_2},$$

with the convention $0 \cdot \infty = 0$ when $A \cap \{f = \alpha_k f\} = \emptyset$. Obviously, $\mathbb{E}_k(b_{A,f}) = 0$ and $\|b_1\|_\infty \leq 1$. Decomposing into level sets as we did in the proof for $p < \infty$, one can show that $\|b_2\|_\infty \leq 4$, details are left to the reader. These L_∞ estimates yield

$$\|b_{A,f}\|_{H_{\text{atb},\infty}^1(\Omega)} \lesssim \mu(A).$$

Moreover, we have

$$\left| \int_{\Omega} f b_{A,f} d\mu \right| = \left| \int_{\Omega} (f - \alpha_k f) b_1 d\mu \right| = \int_A |f - \alpha_k f| d\mu \geq \frac{1}{32} \|f\|_{\text{BMO}}^{\alpha} \mu(A)$$

by assumption (1). This implies

$$\|L_f\|_{H_{\text{atb},\infty}^1(\Omega)^*} \|b_{A,f}\|_{H_{\text{atb},\infty}^1(\Omega)} \geq \frac{1}{32} \|f\|_{\text{BMO}}^{\alpha} \mu(A) \gtrsim \frac{1}{32} \|f\|_{\text{BMO}}^{\alpha} \|b_{A,f}\|_{H_{\text{atb},\infty}^1(\Omega)},$$

which is what we wanted. Finally, if (2) holds there exists $A \in \Sigma_k$ such that

$$\left| \int_A (\alpha_k f - \alpha_{k-1} f) d\mu \right| > \frac{1}{4} \|f\|_{\text{BMO}}^{\alpha}.$$

Let $B = \text{supp}(E_{k-1}(\chi_A)) \in \Sigma_{k-1}$. Define $b_{A,f}$ in this case as

$$b_{A,f} = \chi_A - E_{k-1}(\chi_A).$$

Obviously, it is a ∞ -atomic block. Taking $B_j = \{(j-1)/N < E_{k-1}(\chi_A) \leq j/N\}$, we see that

$$\|b_{A,f}\|_{H_{\text{atb},\infty}^1(\Omega)} \lesssim \mu(A) + \sum_{j=1}^N \|E_{k-1}(\chi_A) \chi_{B_j}\|_{\infty} \mu(B_j),$$

for all N . The sum in the right hand side converges to

$$\int_{\Omega} E_{k-1}(\chi_A) d\mu = \mu(A)$$

as $N \rightarrow \infty$. This shows that $\|b_{A,f}\|_{H_{\text{atb},\infty}^1(\Omega)} \lesssim \mu(A)$. Next we compute

$$\begin{aligned} L_f(b_{A,f}) &= \int_B b_{A,f}(f - \alpha_{k-1} f) d\mu \\ &= \int_A (f - \alpha_{k-1} f) d\mu - \int_B E_{k-1}(\chi_A)(f - \alpha_{k-1} f) d\mu \\ &= \int_A (f - \alpha_k f) d\mu + \int_A (\alpha_k f - \alpha_{k-1} f) d\mu - \int_B E_{k-1}(\chi_A)(f - \alpha_{k-1} f) d\mu. \end{aligned}$$

Since (1) does not hold, we have

$$\left| \int_A (f - \alpha_k f) d\mu \right| \leq \frac{1}{32} \|f\|_{\text{BMO}}^{\alpha} \mu(A).$$

On the other hand, and splitting into level sets we find

$$\begin{aligned} \left| \int_B E_{k-1}(\chi_A)(f - \alpha_{k-1} f) d\mu \right| &\leq \sum_{j=1}^N \frac{j}{N} \left| \int_{B_j} f - \alpha_{k-1} f d\mu \right| \\ &= \sum_{j=1}^N \frac{j}{N} \mu(B_j) \left| \int_{B_j} f - \alpha_{k-1} f d\mu \right| \\ &\leq \sup_{C \in \Sigma_{k-1}} \left| \int_C (f - \alpha_{k-1} f) d\mu \right| \sum_{j=1}^N \frac{j}{N} \mu(B_j) \end{aligned}$$

which is dominated by $\frac{1}{16} \|f\|_{\text{BMO}}^{\alpha} \mu(A)$ for N large enough. So we get

$$\|L_f\|_{H_{\text{atb},\infty}^1(\Omega)^*} \geq \frac{1}{\|b_{A,f}\|_{H_{\text{atb},\infty}^1(\Omega)}} |L_f(b_{A,f})|$$

$$\gtrsim \frac{1}{\mu(A)} \left(\frac{1}{2} - \frac{1}{32} - \frac{1}{16} \right) \|f\|_{\text{BMO}}^\alpha \mu(A) \gtrsim \|f\|_{\text{BMO}}^\alpha.$$

This is the last possible case and completes the proof of Theorem A for $p = \infty$. \square

3. Open problems

When $0 < p < 1$, one can extend the definition of atomic blocks to (p, q) -atomic blocks. Given $1 < q < \infty$, b is called a (p, q) -atomic block when b is Σ_1 -measurable or there exists $k \geq 1$ such that the following properties hold:

- $E_k(b) = 0$,
- $b = \sum_j \lambda_j a_j$ where
 - $\text{supp}(a_j) \subset A_j$,
 - $\|a_j\|_q \leq \mu(A_j)^{1-\frac{1}{p}-\frac{1}{q'}} \frac{1}{k_j-k+1}$,
 for certain $k_j \geq k$ and $A_j \in \Sigma_{k_j}$.

As in the case of $p = 1$, set $|b|_{\text{atb},q}^p = \|b\|_p$ if $b \in L_p(\Omega, \Sigma_1, \mu)$ and

$$|b|_{\text{atb},q}^p = \inf_{\substack{b = \sum_j \lambda_j a_j \\ a_j \text{ } (p,q)\text{-subatom}}} \sum_{j \geq 1} |\lambda_j|$$

otherwise. Finally, we define

$$H_{\text{atb},q}^p(\Omega) = \left\{ f \in L_p(\Omega) \mid f = \sum_i b_i, \text{ } b_i \text{ martingale } (p, q)\text{-atomic block} \right\},$$

equipped with the following quasi-norm

$$\|f\|_{H_{\text{atb},q}^p(\Omega)} = \inf_{\substack{f = \sum_i b_i \\ b_i \text{ } (p,q)\text{-atomic block}}} \left[\sum_{i \geq 1} (|b_i|_{\text{atb},q}^p)^p \right]^{\frac{1}{p}}.$$

The spaces $H_{\text{atb},q}^p(\Omega)$ defined above are quasi-Banach subspaces of $L_p(\Omega)$. One can follow almost *verbatim* the steps in the proof of Theorem A to conclude that the set of linear continuous functionals acting on $H_{\text{atb},q}^p(\Omega)$ can be identified with the Lipschitz type class $\Lambda_{p,q}(\Omega)$ of functions with finite norm

$$\|f\|_{\Lambda_{p,q}(\Omega)} = \sup_{\substack{k \geq 1 \\ A \in \Sigma_k}} \frac{1}{\mu(A)^{\frac{1}{p}-1}} \left[\left(\int_A |f - E_k f|^q d\mu \right)^{\frac{1}{q}} + \|df_k\|_\infty \right].$$

Notice that when $p \rightarrow 1$, the norm in $\Lambda_{p,q}(\Omega)$ tends to the norm in $\text{BMO}(\Omega)$. This motivates our first problem, which is somehow analogous (in the context of atomic blocks of this paper) to Problem 3 in [1].

Problem 3.1. *Do we have*

$$H_{\text{atb},q}^p(\Omega) = H_p(\Omega)$$

for $0 < p < 1$ and $q > 1$? Moreover, do we have $H_p(\Omega)^* = \Lambda_{p,q}(\Omega)$?

Our main result shows that a function in $H_1(\Omega)$ can be decomposed into atomic blocks similar to the ones appearing in the definition of $H_{\text{atb}}^1(\mathbb{R}^n, \mu)$, the atomic block Hardy space of Tolsa [26]. In the proof given in Section 1, we make use of Davis decomposition for martingales. It is natural to ask whether we can find a

description of the space $H_{\text{atb}}^1(\mathbb{R}^n, \mu)$ in terms of some sort of Davis decomposition that splits the space into a (classical) atomic part and a diagonal part. Note that a suitable candidate for the atomic part is the space $h_{\text{at}}^1(\mathbb{R}^n, \mu)$ of functions decomposable into classical atoms supported on doubling sets, since in that case one can easily check that $h_{\text{at}}^1(\mathbb{R}^n, \mu) \subset H_{\text{atb}}^1(\mathbb{R}^n, \mu)$. It is not clear for us what the diagonal part $h_{\text{diag}}^1(\mathbb{R}^n, \mu)$ should be.

Problem 3.2. *Find a diagonal Hardy space*

$$h_{\text{diag}}^1(\mathbb{R}^n, \mu)$$

so that the following Davis type decomposition holds

$$H_{\text{atb}}^1(\mathbb{R}^n, \mu) = h_{\text{at}}^1(\mathbb{R}^n, \mu) + h_{\text{diag}}^1(\mathbb{R}^n, \mu).$$

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